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A PROJECTIVE RELATION AMONG INFINITESIMAL ELEMENTS.

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THEOREM.—If s_1, s_2, s_3 be the lengths of the finitely-distant infinitesimal collinear elements A_1B_1, A_2B_2, A_3B_3; and if s'_1, s'_2, s'_3 be their projections upon another right line by a pencil with any vertex; and if l_1, l_2, l_3 be certain constants, viz: $l_1 = \overline{A_1B_1} \,.\, \overline{A_2B_3}^2$, $l_2 =$ etc.; and if the signs of the radicals be rightly taken, then

$$(1) \qquad \left(\frac{l_1}{s'_1}\right)^{\frac{1}{2}} + \left(\frac{l_2}{s'_2}\right)^{\frac{1}{2}} + \left(\frac{l_3}{s'_3}\right)^{\frac{1}{2}} = 0.$$

So, if $t_1, \ldots t_4$ and $t'_1, \ldots t'_4$, be the areas of the finitely-distant infinitesimal complanar triangles $A_1B_1C_1, \ldots A_4B_4C_4$ and of the projections, and if their constants $m_1 = t_1 \,.\, (\text{area } \overline{A_2A_3A_4})^3$, $m_2 =$ etc., then

$$(2) \qquad \left(\frac{m_1}{t'_1}\right)^{\frac{1}{3}} + \ldots + \left(\frac{m_4}{t'_4}\right)^{\frac{1}{3}} = 0.$$

So, if $v_1, \ldots v_5$ and $v'_1, \ldots v'_5$ be the volumes of tetrahedrons and of their projections, then

$$(3) \qquad \left(\frac{n_1}{v'_1}\right)^{\frac{1}{4}} + \ldots + \left(\frac{n_5}{v'_5}\right)^{\frac{1}{4}} = 0,$$

where the constants $n_1 = v_1 \,.\, (\text{vol. } \overline{A_2A_3A_4A_5})^4$, $n_2 =$ etc.

And similarly for "homaloidal space" of 4, 5, or more dimensions.

The triangular, tetrahedral, etc., elements in (2), (3), etc., are taken infinitesimal in all their dimensions, and are evidently replaceable by any other elements that are so; e. g., by infinitesimal circles and spheres deforming into ellipses and ellipsoids.

PROOF OF (1). The anharmonic ratio

$$\frac{-A'_2B'_3 \,.\, B'_2A'_3}{A'_2B'_2 \,.\, A'_3B'_3} = \frac{-A_2B_3 \,.\, B_2A_3}{A_2B_2 \,.\, A_3B_3};$$

or in the limit, when points A coincide with points B,

$$\overline{A'_2A'_3}^2 = \frac{s'_2\, s'_3}{s_2\, s_3} \,.\, \overline{A_2A_3}^2 = \frac{s'_1 s'_2 s'_3}{s_1 s_2 s_3} \,.\, \frac{l_1}{s'_1}$$

So,

$$\overline{A'_3A'_1}^2 \text{ and } \overline{A'_1A'_2}^2 = \text{etc., by symmetry.}$$

But

$$\overline{A'_2A'_3} + \overline{A'_3A'_1} + \overline{A'_1A'_2} = 0,$$

wherein, as in Tait's Quaternions, the vinculums show that the segments are measured in the directions indicated.

Hence the formula (1).

The proofs of (2), (3), etc., are similar; e. g., it is known that the ratio of products of triangle-areas

$$\frac{\overline{A'_2B'_3C'_4}\cdot\overline{B'_2C'_3A'_4}\cdot\overline{C'_2A'_3B'_4}}{\overline{A'_2B'_2C'_2}\cdot\overline{A'_3B'_3C'_3}\cdot\overline{A'_4B'_4C'_4}}=\frac{\overline{A_2B_3C_4}\cdot\overline{B_2C_3A_4}\cdot\overline{C_2A_3B_4}}{\overline{A_2B_2C_2}\cdot\overline{A_3B_3C_3}\cdot\overline{A_4B_4C_4}},$$

because each member is a function of only the angles made at the centre of projection by the different projecting rays; hence, in the limit,

$$\overline{A'_2A'_3A'_4}^{\,3}=\frac{t'_1\;.\;.\;.\;t'_4}{t_1\;.\;.\;.\;t_4}\cdot\frac{m_1^3}{t'_1}\text{, etc.,}$$

where the vinculum shows that the area is taken positive or negative according as the cyclic order of the vertices corresponds to positive or negative rotation. But

$$\overline{A'_2A'_3A'_4}+\overline{A'_1A'_4A'_3}+\overline{A'_4A'_1A'_2}+\overline{A'_3A'_2A'_1}=0,$$

which gives (2).

The rule of signs for linear, triangular, pyramidal, and other elements is like that used in determinants, viz: every interchange of two letters under the vinculum changes the value of the symbol from positive to negative or the reverse. When no vinculum is used, the length or area, etc. indicated is taken positive, or rather signless.

Since the location of the centre of projection, and of $A'_1B'_1\;.\;.\;.\;A'_2\;.\;.\;.$, always requires one dimension beside those occupied by $A_1B_1\;.\;.\;.\;A_2\;.\;.\;.$, the proof and even the statement of (3) is 4-dimensional. We might avoid this by defining $A'_1B'_1\;.\;.\;.\;D'_5$ as any figure homographic to $A_1B_1\;.\;.\;.\;D_5$, and by using this principle, that if in two homographic figures coresponding tetrahedral volumes be each divided by the product of its vertices' distances from the centre of homology, then the quotients have a fixed ratio. A like treatment would apply to (2), etc. Indeed, homographic figures are only projected figures so moved without further distortion as to be collinear, complanar, or co-spatial, etc. But projection, in however many dimensions, appears to me the more natural procedure.